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EFFECTS OF DAMPING CONSTANTS AND STRESS DISTRIBUTION ON THE RESONANCE RESPONSE OF MEMBERS

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WRIGHT AIR DEVELOPMENT CENTER

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Air Research and Development Command
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Wright-Patterson Air Force Base, Ohio

FOREWORD

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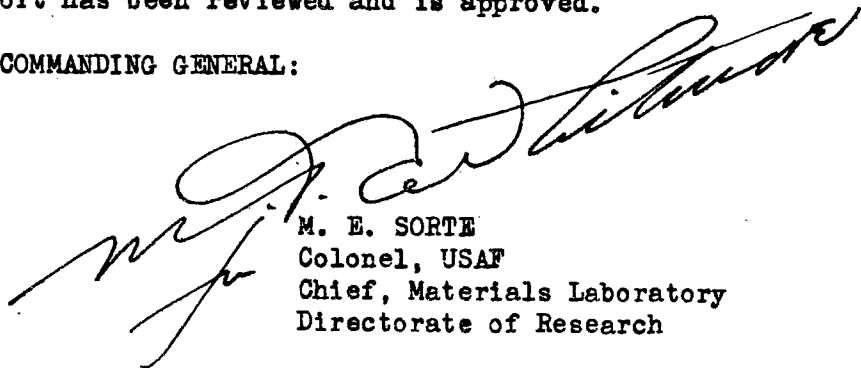Mr. W. J. Trapp acting as project engineer.

ABSTRACT

The amplitude of vibration of a member at resonance, as defined by its resonance amplification factor, is analyzed in relationship to the damping properties of materials. Data are presented on damping energy to indicate the effect of stress magnitude, stress history and temperature. Based on the mathematical relationship found to exist between damping and stress magnitude the resonance amplification factors are determined for a variety of direct stress members and beams. It is shown that the amplification in vibration caused by resonance may be considered to be the product of three basic factors: (a) the mathematical factor, (b) the cross-sectional shape factor, and (c) the longitudinal stress-distribution factor. The first of these factors may be calculated from the damping and dynamic modulus properties of the material and the last two from the shape and loading characteristics of the member. Diagrams are presented to show these basic factors as functions of the damping exponent and other variables for members commonly encountered in engineering practice. Experimental data are presented to confirm the equations derived for resonance amplification factors of members having various shapes and stress distribution.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDING GENERAL:

M. E. SORTE
Colonel, USAF
Chief, Materials Laboratory
Directorate of Research

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Effect of Damping Constants and Stress Distribution on the Resonance Response of Members

By B. J. LAZAN,[1] MINNEAPOLIS, MINN.

INTRODUCTION

NEAR-resonance vibration is generally considered to be a common cause for fatigue and other types of service failures in many and diverse fields of engineering. Even if actual failure can be avoided, the rough and noisy operation associated with the near-resonance condition frequently necessitates correction. Current trends toward higher speeds and decreased factor of safety have increased the importance of resonance vibration as a factor in design.

Considerable work has been done on the fatigue and other properties of materials which define their ability to withstand cyclic stress produced by resonance or other vibration conditions. However, relatively little has been done on the analysis of the properties of materials and other factors which govern the amplification in vibration that is caused by the near-resonance condition. Hence this paper is concerned with an analysis of factors which govern the relationship of the external force which excites a vibration (hereafter called the exciting force) to the internal force excited in the member at resonance (hereafter called the resonance excited force). The ratio of excited to exciting force shall be called (1)[2] the resonance amplification factor A_r, in this paper.

The general nature of the viscous damped resonance curve is well known (2) and will not be discussed here. Although the internal hysteresis damping of a material is fundamentally different in nature from viscous damping (3), and the resonance curves may be quite unsymmetrical (4), the resonance amplification factor nevertheless provides a convenient measure of the severity of the resonant condition. Consequently, this factor shall be used in this paper as a basis for analysis.

VARIABLES WHICH AFFECT MAGNITUDE OF DAMPING ENERGY IN MATERIALS AND PARTS

The energy absorption by an actual part, or its damping, may be due to several factors, among which are: (a) The inelastic behavior of the material as indicated by its damping capacity, (b)

slippage and other structural or joint factors, and (c) aerodynamic effects. In general, there is very little data on the relative magnitude and importance of each of these three absorbers of damping energy. It is likely that in many applications the primary absorber of damping energy is the structure or joint factor and the contributions offered by the material damping are insignificant. However, there is little doubt that to generalize this statement, which is sometimes done, is highly misleading. In many applications subjected to resonance vibrations it is difficult to include significant structural damping and one must rely on material damping. Even where significant structural and aerodynamic damping may be present it is shown by recent work (5) that material damping may still be sufficiently large to be highly significant as a limiter of resonance vibrations.

Since this paper is concerned primarily with the dynamic properties of materials, structural and aerodynamic damping shall not be discussed further and only the material damping factor will be analyzed in detail.

With this as background let us now discuss, as an indication of the variables which affect the magnitude of damping in materials, the dynamic properties of one particular material. For this discussion, data on the temperature-resistant material S-816 at three temperatures will be presented so as to illustrate a variety of trends and patterns of significance in resonance-vibration studies.

For brevity, the recently developed damping, elasticity, and fatigue testing machines (3, 6) used to procure the data on S-816 will not be described in this paper. Also, for brevity, diagrams showing the variation in damping energy absorbed by the material at various stress magnitudes as a function of a number of cycles of fatigue stress (5) are omitted. However, in order to establish mathematical relationships for later analysis, damping energy is plotted as a function of stress magnitude in Fig. 1, the different curves indicating the relationship for different stress histories and temperatures. The significance of the right-hand

[2] Numbers in parentheses refer to the Bibliography at the end of the paper.
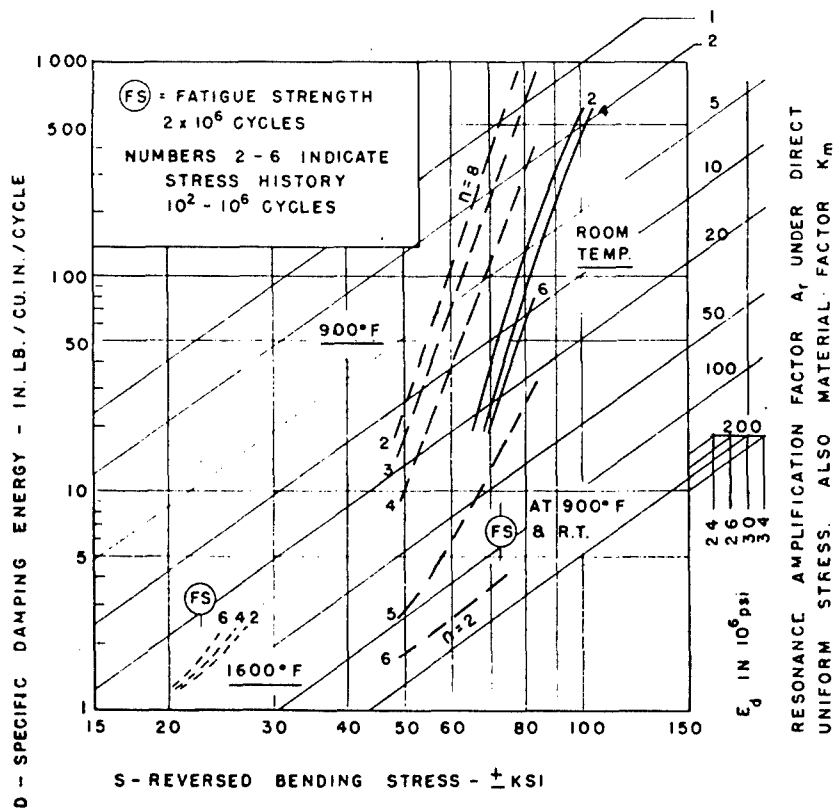
Fig. 1 D-S-N-A_r Diagram for Alloy S-816 Indicating Effect of Stress Magnitude
and Stress History on Damping Energy

ordinate of this figure will be discussed later. The series of curves shown at each temperature (solid lines represent room temperature, long dash 900 F, and short dash 1600 F) are identified with the numbers 2, 3, 4, 5, and 6 to indicate the damping values after 10^2, 10^3, 10^4, 10^5, and 10^6 cycles of stress. The stress level during the imposition of the stress history was the same as during the damping measurement. The flags enclosing the letters F.S. indicate the fatigue strengths of the material.

Due to space limitations, data for S-816 only are presented in this paper. It should be pointed out, however, that the damping values do not always decrease with an increase in number of cycles at room temperature as shown in Fig. 1. Many different patterns of behavior have been observed; for example, the damping of such powdered metals increases continuously with an increase in the number of cycles; 403 stainless displays an initial decrease followed by an increase to a peak after which the values decrease if the specimen does not previously fracture. The data for some metals show damping values that decrease to a minimum, followed by a steady increase to fracture; other materials display practically no change in damping with number of cycles; and Inconel X has high initial damping followed by a period of practically no change, after which the damping increases sharply preceding failure. The general trends for a given material may be the same at elevated temperature as at room temperature or they may be reversed, as in the case of S-816 at room temperature and 1600 F. In most cases studied to date, however, the plot of log damping versus log stress (at constant stress history and constant temperature) may be represented by reasonably straight lines.

MATHEMATICAL RELATIONSHIP BETWEEN DAMPING AND STRESS MAGNITUDE

It is desirable at this point to determine the mathematical

relationships between damping and stress magnitude, stress history, and temperature. As indicated previously, stress history affects different materials in pronouncedly different manners. It is therefore impossible to generalize stress-history effects at this time and express such effects mathematically. It is also impossible to generalize temperature effects at this time. However, there are sufficient data (7, 8, 5) to indicate that for many materials the effect of stress magnitude (at constant temperature and stress history) may be expressed by the equation

$$D = J S^n \dots\dots\dots\dots\dots\dots [1]$$

where

D = specific damping energy, in-lb/cu in/cycle

S = stress, psi

J, n = const

Referring to Fig. 1, it may be seen that the log D versus log S is reasonably straight with slopes ranging from 2 to 8. Thus for S-816, exponent n varies between 2 and 8 depending on stress history, magnitude, and temperature.

In some of the earlier work (4, 8) exponent n was found to be approximately 3 for several materials at room temperature. However, most of the materials now under study at the University of Minnesota display exponents n which generally are greater than 3, particularly if various stress histories are included. An exponent n as large as 30 has been observed for one material at high stress.

In future work more careful analyses will be undertaken of stress history and temperature effects. For a given material it may be possible to handle these variables mathematically by considering J and n as functions of stress history and temperature, rather than as constants as assumed in the foregoing.

Resonance Amplification Factor Under Uniform Direct Stress

So far basic data only have been presented on the damping energy. Now what does such data mean in terms of vibration amplification of actual parts at resonance?

The effect of damping energy on the resonant behavior of a part may be specified in terms of the resonance amplification factor A_r, as briefly discussed previously. This factor is the ratio of the force which is excited during resonance vibration to the force exciting the vibration. Thus A_r is a measure of the severity of a resonant condition. For most applications in which linearity may be assumed with reasonable accuracy this factor generally may be computed (1) from the equation

$$A_r = 2\pi W_0/D_0 \dots\dots\dots\dots [2]$$

where

W_0 = elastic energy in member at maximum stress, in-lb
D_0 = total damping energy in in-lb/cycle, VD
V = volume of material at stress

Therefore, for the special case of direct uniform stress for which the elastic energy $W_0 = VS^2/2E_d$

$$A_r = \pi S^2/E_d\,D = \pi/E_d J S^{n-2} \dots\dots\dots\dots [3]$$

where E_d = dynamic modulus of elasticity (5), psi.

If the modulus of elasticity E_d is assumed to be reasonably constant, then A_r is a function of the variables S and D only, and for this simplified case of direct uniform stress only, a grid of lines may be drawn in Fig. 1 as shown (see right-hand ordinate) to indicate the values of A_r. However, as shown in a previous publication (5) the modulus of elasticity does not remain constant but varies between 21 and 34 × 10⁶ psi, depending on temperature, stress magnitude, and stress history. Therefore the right-hand scale in Fig. 3 includes provisions for determining A_r for various values of E_d as indicated at the lower end of this scale.

The wide range of A_r values observed for S-816, as revealed by Fig. 1, is of considerable engineering significance. Depending upon stress magnitude and stress history, A_r under direct uniform stress may be insignificantly small or as large as 150. At 900 F, for example, stress history alone may increase A_r to over 100 times its virgin value. It is therefore apparent that the specification of A_r for a given material and its comparison with other materials cannot be done in a simple manner, since two variables other than material and temperature are involved.

It is desirable to review for future reference the relationship between damping exponent n and change in A_r under increasing stress. As indicated previously, if $n = 2$ then A_r is independent of stress magnitude since both elastic and damping energy increase as the square of stress. If, however, n is larger than 2, which is the usual case, the damping energy increases more rapidly with stress than does the elastic energy resulting in a A_r which decreases with increasing stress. These observations will be amplified in the analysis of cross-section shape and longitudinal stress-distribution effects in later sections.

Resonance Amplification Factor Under Nonuniform Direct Stress

All discussion to this point has been concerned primarily with the basic damping, elasticity, and fatigue properties of materials and the interpretation of these properties in terms of behavior of a resonant member under uniform direct stress only. In practice this rather desirable state of uniform stress is, of course, quite rare. Generally speaking, critical members are almost always stressed nonuniformly, in which case the equation $A_r = \pi S^2/E_d\,D$ does not hold. Therefore, in the interest of providing wider engineering applicability for the type of data presented previously, this section is concerned with an analysis of the resonance amplification factor A_r under nonuniform direct stress. Bending members are analyzed in the next section. Other types of members, such as those subjected to torsional stress, are not included in this paper for reasons of brevity, but may be analyzed in a similar manner.

Analyses of members under resonant vibrations have been made in the past (9, 8, 1) to indicate that fatigue strength is not the sole criterion for indicating resonance durability. However, an attempt is made in this section to undertake a more thorough treatment of the important variables and to generalize the equations expressing the resonance amplification factor of members. This type of analysis is also given in reference (10) published after the completion of this paper.

The general case of direct stress vibration covered in this section considers the inertia forces produced by and within the vibrating member. The force on a section of a member with internal inertia forces will be, of course, a function of the longitudinal location of the section. Although the member may have variable cross section as shown in the diagram in Table 1, it is assumed that the stress at a given cross section is uniformly distributed (although the total force on the section and the resultant stress may vary along the length of the member).

As indicated in Equation [2] the resonance amplification factor A_r for a member generally may be computed from the equation $A_r = 2\pi W_0/D_0$.

Referring to the figure and symbol definitions in Table 1 the total elastic energy

$$W_0 = \int dW_0 = \int W_x\,dV = \int_0^a \frac{S_x^2}{2\,E_d}\,A_x\,dx \dots\dots [4]$$

where W_x is the unit elastic energy at any location and dV is a small differential volume having thickness dx.

As indicated (5), some materials display sufficiently large variation in E_d to necessitate considering this modulus a function of stress level and stress history. However, for most materials and stress ranges, E_d may be assumed to be a constant for most engineering calculations. Therefore E_d may be moved from within the integral sign as indicated

$$W_0 = \frac{1}{2E_d}\int_0^a S_x^2\,A_x\,dx \dots\dots\dots\dots [5]$$

Again, referring to the figure and symbols in Table 1, the total damping energy

$$D_0 = \int dD_0 = \int D_x\,dV = J\int_0^a S_x^n\,A_x\,dx \dots\dots [6]$$

Combining Equations [2], [5], and [6]

$$A_r = \frac{\pi}{E_d J}\frac{\int_0^a S_x^2\,A_x\,dx}{\int_0^a S_x^n\,A_x\,dx} = \frac{\pi}{E_d J S_m^{n-2}}\frac{\int_0^a \left(\frac{S_x}{S_m}\right)^2 A_x\,dx}{\int_0^a \left(\frac{S_x}{S_m}\right)^n A_x\,dx} \dots [7]$$
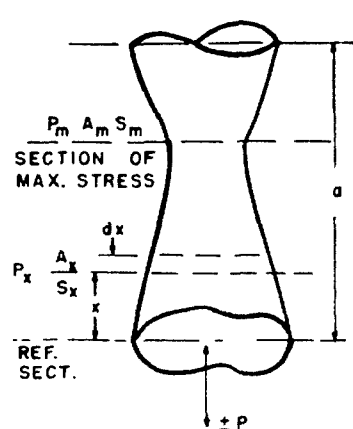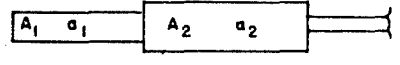
In the foregoing equation, S_m is the stress at the section of maximum stress, as defined in Table 1.

For convenience in analysis and interpretation, it is desirable to define the following factors (justification for these definitions will appear later)

$$\text{Material factor, } K_m = \frac{\pi}{E_d J S_m^{n-2}} \dots\dots\dots\dots [8]$$

$$\text{Longitudinal stress-distribution factor, } K_s = \frac{\int_0^a \left(\frac{S_x}{S_m}\right)^2 A_x\,dx}{\int_0^a \left(\frac{S_x}{S_m}\right)^n A_x\,dx} \dots\dots\dots\dots [9]$$

TABLE I:- SUMMARY OF VALUES FOR RESONANCE AMPLIFICATION FACTOR A_r UNDER DIRECT STRESS

$A_r = K_m K_c \cdot K_s$, WHERE $K_m = \pi/E_d \sqrt{S_m^{n-2}}$, $K_c = 1$, AND K_s GIVEN BELOW IN SEVERAL CASES

	LONGITUDINAL SHAPE AND STRESS DISTRIBUTION IN MEMBER	VALUES OF K_s	DEFINITION OF SYMBOLS
I	GENERAL CASE OF DIRECT STRESS FOR EITHER (1) CONTINUOUS MEMBERS OR (2) MEMBERS WITH DISCONTINUITIES. CONSIDERING INTERNAL INERTIA FORCES.	$\dfrac{\int_0^a (S_x/S_m)^2 A_x\,dx}{\int_0^a (S_x/S_m)^n A_x\,dx}$	RESONANCE EXCITING FORCE = P_o. EXCITED FORCE AT SECTION A_m OF MAX. STRESS $S_m = P_m = P_o A_r$
IA	CONTINUOUS MEMBERS HAVING CONSTANT CROSS-SECTIONAL AREA. $A_x = A_m$ IN GENERAL, $S_x \neq P_m/A_x$ IF INTERNAL INERTIA FORCES ARE SIGNIFICANT	$\dfrac{\int_0^a (S_x/S_m)^2\,dx}{\int_0^a (S_x/S_m)^n\,dx}$	
IA1	INTERNAL INERTIA FORCES NEGLIGIBLE $S_x = P_m/A_m = S_m$	$K_s = 1$	
IA2	INTERNAL INERTIA FORCES SUCH THAT: $S_x = (x/a)S_m$	$(n+1)/3$	
IA3	INTERNAL INERTIA FORCES SUCH THAT: $S_x = (x/a)^m S_m$	$(mn+1)/(2m+1)$	
IA4	INTERNAL INERTIA FORCES SUCH THAT: $S_x = S_m \sin(\pi x/2a)$ (LONG. VIB. IN CANTILEVER ROD)	$\pi/4 \int_0^{\pi/2} \sin^n \theta\, d\theta$	FORCE ON REFERENCE SECTION
IB	CONTINUOUS MEMBERS HAVING VARIABLE AREA	USE EQUATION I	IF INTERNAL INERTIA FORCES ARE NEGLIGIBLE: THEN $P = P_m$
IB1	INTERNAL INERTIA FORCES NEGLIGIBLE $S_x = P_m/A_x$	$A_m^{2-n} \dfrac{\int_0^a A_x^{-1}\,dx}{\int_0^a A_x^{1-n}\,dx}$	IF EXTERNAL INERTIA FORCES ARE NEGLIGIBLE: THEN $P = P_o$
IB1a	ROD HAVING TAPER "s" $A_x = A_m[1+(x/a)(s-1)]$	$\dfrac{(n-2)\log s}{(1-s^{2-n})}$	
IB2	INTERNAL INERTIA FORCES SUCH THAT: $S = S_m(x/a), S_m \sin(\pi x/2a)$, ETC.	USE EQUATION I	
IC	DISCONTINUOUS MEMBER HAVING CONSTANT CROSS-SECTION BETWEEN DISCONTINUITIES. $A_1\ a_1 \quad A_2\ a_2$	$\dfrac{A_1\int_0^{a_1}(S_x/S_m)^2\,dx + A_2\int_{a_1}^{a_2+a_1}(S_x/S_m)^2\,dx + ---}{A_1\int_0^{a_1}(S_x/S_m)^n\,dx + A_2\int_{a_1}^{a_2+a_1}(S_x/S_m)^n\,dx + ---}$	
IC1	INTERNAL FORCES NEGLIGIBLE $S_x = P_m/A_x$	$\dfrac{(a_1/a_2)+(A_1+A_2)}{(a_1/a_2)+(A_1+A_2)^{n-1}}$	FOR TWO SECTION MEMBER (ONE DISCONTINUITY) $A_m = A_1$ & $a_m = a_1$
IC2	INT. INERTIA FORCES NOT NEGLIGIBLE.	USE EQUATION IC.	
ID	DISCONTINUOUS MEMBERS WITH VARIABLE SECTION BETWEEN DISCONTINUITIES.	USE EQUATION I.	

In accordance with these definitions, the resonance amplification factor for the general case of direct stress may be written as

$$A_r = K_m K_s \dots \dots \dots \dots \dots [10]$$

The material factor K_m is, of course, independent of cross-sectional shape and stress distribution of the member; it depends only on the properties of the material E_d, J, and n, and the operating stress S_m. Referring to Equation [3], it may be seen that the material factor is identical to the resonance amplification factor A_r under direct uniform stress. This relationship will be discussed again in connection with bending members.

The longitudinal stress distribution factor K_s, separated from K_m as shown in Equations [7], [9], and [10], is a function of member shape (as specified by A_x), longitudinal stress distribution S_x, member length a, and damping exponent n, and it is independent of damping constant J.

In the analysis of bending members in a later section it will be observed that A_r also may be separated into material and longitudinal stress-distribution factors K_m and K_s, with an additional K_c-factor required to account for cross-sectional stress distribution. In the interest of generalizing, this K_c-factor is included in the summary for direct stress members given in Table 1, K_c being equal to 1 for this case since a uniform stress distribution at a given cross section is assumed.

Referring to Table 1, the first row lists the general case of direct stress and subsequent rows cover various special direct-stress cases encountered in engineering practice. For each of the general, semigeneral, and special cases listed, the values of K_s are given in the second column. For conciseness, the derivations for the K_s-equations listed in this table are not shown in this paper. In general, these equations are rather simple mathematical reductions of the equation for general case I with appropriate substitutions for S_x and A_x. The trends revealed by these K_s-equations will be discussed later.

RESONANCE AMPLIFICATION FACTOR UNDER BENDING STRESS

An analysis of the resonance amplification factor of beams, a structural component in which resonant vibrations are often of critical importance, is given in the following.

Referring to the figures in Table 2, the elastic and damping energy of a beam of general shape may be expressed as indicated in the following

Total elastic energy, $W_0 = \int 1/2\, M_x\, d\theta = \dfrac{1}{2}\int_0^a \dfrac{M_x^2\, dx}{E_d I_x}$

and assuming E_d is constant as discussed previously

$$W_0 = \dfrac{1}{2 E_d}\int_0^a \dfrac{S_x^2 I_x\, dx}{t_x^2} \dots \dots \dots \dots \dots [11]$$

Total damping energy $= D_0 = \int D\, dv = \int\int DZ\, dy\, dx$

but

$$D = J S^n \text{ and } S = \dfrac{y}{t_x} S_x$$

where S = stress at any point, S_x = maximum stress (at outer fibers) at any position x, and t_x = maximum distance from neutral axis to outer fibers at any position x.

Thus, since t_x and S_x are not functions of y

$$D_0 = \int\int J \dfrac{y^n}{t_x^n} S_x^n Z\, dy\, dx$$
$$= J \int_{x=0}^a \left[\int_{y=-t'(x)}^{y=t(x)} y^n Z\, dy \right] \dfrac{S_x^n}{t_x^n} dx \dots \dots [12]$$

where $t(x)$ and $t'(x)$ are functions of x which define the location of the fibers most distant from neutral axis of the beam.

Combining Equations [2], [11], and [12]

$$A_r = \dfrac{2\pi}{2 E_d J}\; \dfrac{\displaystyle\int_0^a S_x^2 \dfrac{I_x}{t_x^2}\, dx}{\displaystyle\int_{x=0}^a \left[\int_{y=-t'(x)}^{t(x)} y^n Z\, dy \right] \dfrac{S_x^n}{t_x^n}\, dx}$$

$$= \dfrac{\pi}{E_d J S_m^{n-2}}\; \dfrac{\displaystyle\int_0^a \left(\dfrac{S_x}{S_m}\right)^2 \dfrac{I_x}{t_x^2}\, dx}{\displaystyle\int_{x=0}^a \left[\int_{y=-t'(x)}^{t(x)} y^n Z\, dy \right] \left(\dfrac{S_x}{S_m}\right)^n t_x^{-n}\, dx}$$

$$\dots \dots \dots \dots \dots [13]$$

Inspecting Equation [13], it is apparent that the first factor $\pi / E_d J S_m^{n-2}$ is the material factor K_m discussed previously. It will be shown that, in general, the remaining part of this equation may be considered to be the product of two other factors: (a) The longitudinal stress distribution factor K_s, as discussed previously, and (b) the cross-sectional shape factor K_c to be discussed later.

In the foregoing general case these two factors cannot be separated conveniently and only their product ($K_s K_c$) is specified. However, in most special cases discussed below, K_s and K_c can be separated.

Consider now the special case of beams having constant cross-sectional moment of inertia I and neutral-axis distance t and t' within the length of the beam under consideration. Under these conditions the integral

$$\int_{y=t'(x)}^{t(x)} y^n Z\, dy$$

reduces to

$$\int_{-t'}^{t} y^n Z\, dy$$

which is not a function of x and may therefore be removed from within the $\int_{x=0}^a dx$ integral. Thus Equation [13] reduces to

$$A_r = \dfrac{\pi}{E_d J S_m^{n-2}} \cdot \dfrac{\displaystyle\int_0^a \left(\dfrac{S_x}{S_m}\right)^2 dx}{\displaystyle\int_0^a \left(\dfrac{S_x}{S_m}\right)^n dx} \cdot \dfrac{I\, t^{n-2}}{\displaystyle\int_{-t'}^{t} y^n Z\, dy} \dots [14]$$

Comparing Equation [14] with direct-stress Equation [7] (with A_x = const), it may be seen that the material factor

$$K_m = \dfrac{\pi}{E_d J S_m^{n-2}}$$

and the longitudinal stress-distribution factor

$$K_s = \dfrac{\displaystyle\int_0^a \left(\dfrac{S_x}{S_m}\right)^2 dx}{\displaystyle\int_0^a \left(\dfrac{S_x}{S_m}\right)^n dx}$$

are identical to those defined in Equations [8] and [9]. All that remains to be accounted for in Equation [14] is therefore the last factor, which may be defined as follows

$$\text{Cross-sectional shape factor } K_c = \dfrac{I\, t^{n-2}}{\displaystyle\int_{-t'}^{t} y^n Z\, dy} \dots [15]$$

TABLE II:- SUMMARY OF VALUES FOR RESONANCE AMPLIFICATION FACTOR A_r FOR BEAMS

$A_r = K_m \cdot K_c \cdot K_s$, WHERE $K_m = \pi/E_d J S_m^{n-2}$, AND K_c AND K_s ARE GIVEN BELOW FOR SEVERAL CASES

LONGITUDINAL SHAPE AND STRESS DISTRIBUTION	VALUES FOR K_c, K_s OR $K_s \cdot K_c$		SECTION OF MAX STRESS
II. GENERAL CASE OF BENDING STRESS FOR EITHER (1) CONTINUOUS BEAMS OR (2) BEAMS WITH DISCONT. WITH SIGNIFICANT INTERNAL INERTIA FORCES CONSIDERED.	$K_c \cdot K_s = \dfrac{\displaystyle\int_0^a (S_x/S_m)^2 (I_x/t_x^2)\,dx}{\displaystyle\int_{x=0}^a \int_{y=t'(x)}^{t(x)} y^n z\,dy\,\dfrac{(S_x/S_m)^n}{t_x^n}\,dx}$		
IIA. CONTINUOUS BEAM HAVING CONSTANT CROSS-SECTION. $I_x = I$ $t_x = t$	$K_c = \dfrac{I\,t^{n-2}}{\displaystyle\int_{-t'}^{t} y^n z\,dy}$ $\quad K_s = \dfrac{\displaystyle\int_0^a (S_x/S_m)^2 dx}{\displaystyle\int_0^a (S_x/S_m)^n dx}$		

	SPECIAL CROSS-SECTIONAL SHAPE CONSIDERED VALUES OF $K_s \cdot K_c$	RECTANGLE $K_c = 0.33(n+1)$	CIRCLE $K_c = \dfrac{0.196(n+2)}{\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta}$	DIAMOND $K_c = 0.083(n+1)(n+2)$
IIA1	CONSTANT MOMENT $S_x = S_m$; $K_s = 1$	$0.33(n+1)$	$\dfrac{0.196(n+2)}{\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta}$	$0.083(n+1)(n+2)$
IIA2	VARIABLE MOMENT SUCH THAT $S_x = (x/a)S_m$ $K_s = (n+1)/3$	$0.111(n+1)^2$	$\dfrac{0.065(n+1)(n+2)}{\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta}$	$0.028(n+1)^2(n+2)$
IIA3	VARIABLE MOMENT SUCH THAT $S_x = (x/a)^m S_m$ $K_s = (mn+1)/(2m+1)$	$\dfrac{0.33(n+1)(mn+1)}{2m+1}$	$\dfrac{0.196(mn+1)(n+2)}{(2m+1)\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta}$	$\dfrac{0.083(n+1)(n+2)(mn+1)}{(2m+1)}$
IIA4	VARIABLE MOMENT SUCH THAT $S_x = S_m \sin(\pi x/2a)$ $K_s = \pi/4 \displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta$	$\dfrac{0.26(n+1)}{\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta}$	$\dfrac{0.154(n+2)}{\left[\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta\right]^2}$	$\dfrac{0.065(n+1)(n+2)}{\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta}$

	DISCONTINUOUS BEAMS HAVING CONSTANT CROSS-SECTION BETWEEN DISCONTINUITIES		
IIB	$t_1\,a_1\,I_1$ $t_2\,a_2\,I_2$ $t_g\,a_g\,I_g$	$K_c = \dfrac{I_1 t_1^{n-2}}{\displaystyle\int_{-t'}^{t} z\,y^n dy}$ $\quad K_s = \dfrac{1 + \left[\dfrac{t_1}{t_2}\right]^2\left[\dfrac{a_1}{a_2}\right]\left[\dfrac{s_2}{s_1}\right]^2\left[\dfrac{I_2}{I_1}\right] + \cdots + \left[\dfrac{t_1}{t_g}\right]^2\left[\dfrac{a_g}{a_1}\right]\left[\dfrac{s_g}{s_1}\right]^2\left[\dfrac{I_g}{I_1}\right]}{1 + \left[\dfrac{t_1}{t_2}\right]^n\left[\dfrac{a_1}{a_2}\right]\left[\dfrac{s_2}{s_1}\right]^n \displaystyle\int_{-t_2'}^{t_2} z\,y^n dy \Big/ \displaystyle\int_{-t_1'}^{t_1} z\,y^n dy + \cdots}$		

	CONSTANT MOMENT ON TWO SECTION MEMBER (1 SIMILAR TO 2)	RECTANGLE	CIRCLE	DIAMOND
IIB1	$K_s = \dfrac{\dfrac{a_1}{a_2} + \dfrac{I_1}{I_2}}{\dfrac{a_1}{a_2} + \left[\dfrac{I_1}{I_2}\right]^{\frac{3n-2}{4}}}$	$\dfrac{0.33(n+1)\left[\dfrac{a_1}{a_2}+\dfrac{I_1}{I_2}\right]}{\dfrac{a_1}{a_2}+\left[\dfrac{I_1}{I_2}\right]^{\frac{3n-2}{4}}}$	$\dfrac{0.196(n+2)\left[\dfrac{a_1}{a_2}+\dfrac{I_1}{I_2}\right]}{\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta\left\{\dfrac{a_1}{a_2}+\left[\dfrac{I_1}{I_2}\right]^{\frac{3n-2}{4}}\right\}}$	$\dfrac{(n+1)(n+2)\left[\dfrac{a_1}{a_2}+\dfrac{I_1}{I_2}\right]}{12\left\{\dfrac{a_1}{a_2}+\left[\dfrac{I_1}{I_2}\right]^{\frac{3n-2}{4}}\right\}}$

Inspection of Equation [15] will show that the name assigned to this factor is descriptive of its nature.

Therefore it appears to be possible to define the resonance amplification factor of beam by the equation

$$A_r = K_m K_s K_c \dots \dots \dots \dots \dots [16]$$

In all cases K_m appears as a separate factor. Although for the case of beams of uniform cross section and other special cases it may be possible to separate the K_s and K_c-factors as shown, in the most general case only the product $K_c K_s$ can be specified.

Table 2 is a summary of values for the resonance amplification factor A_r for various general and special types of beams encountered in engineering practice. The arrangement of this table is similar to that of Table 1 and is self-explanatory. It should be noted that three cross-sectional shapes frequently encountered in engineering practice (the rectangle or square, circle, and diamond) are analyzed merely to indicate trends to be discussed later. Other shapes can, of course, also be analyzed by the same methods.

From Table 2 it may be noted that the equation for a vibrating beam (case IIA2) of rectangular cross section is

$$A_r = \frac{\pi}{E_d S^{n-2}} \left(\frac{n+1}{3} \right)^2$$

If this equation is used to compute the allowable force P_0 at resonance and if a value of 3 is substituted for n, then the resultant equation is in exact agreement with Marin's Equation [14] in reference (9).

Referring to Equation [14] of this paper, it may be seen that both the K_s and K_c-factors are functions of damping exponent n. Since this exponent is widely different for different materials, a range from 2 to 30 having been observed to date, it is desirable to diagram these factors as a function of n as shown in Figs. 2 and 3. These diagrams also reveal trends which provide a basis for discussing the interpretation of damping data in terms of resonance amplification factor as used in engineering design.

Fig. 2 diagrams the longitudinal stress-distribution factor K_s, as a function of damping exponent n for various common shapes of members and types of loading. Members without discontinuities are shown in solid lines and members with one discontinuity (two section members) are shown in broken lines.

For all the cases referred to in Fig. 2, factor K_s equals 1 at $n = 2$, and, except for uniform direct stress, increases with increasing values of n. The rate of increase in K_s with n is greatly dependent on the type of member and loading.

For example, for a freely vibrating cantilever member of constant cross section under either tension or bending stress (stress distribution assumed to be $S_x = S_m \sin[\pi x/2a]$ as indicated for cases IA4 and IIA4) K_s increases from 1 to only 3.5 as damping exponent n increases from 1 to 30. As a second example, K_s for a tapered tension member (case IB1a) having a taper s of 4 displays the much larger increase from 1 to 40 in the same range of n-values.

In the case of discontinuous members (broken lines), it is of course possible to have an infinite number of ratios of section area, and so on, at the discontinuities. For simplicity, members with one discontinuity only are considered, and for further simplification only a few ratios q of areas A_1 to A_2 and only a few ratios p of lengths a_1 to a_2 between discontinuities are included. Small values of ratio p were selected to indicate the characteristics of members with notches and other stress-concentration effects. For example, a member with a small notch (which allows relatively low working stress in the rest of the member) can attain K_s-values as high as 80. This means that at the same nominal working stress a notched member may have 80 times the
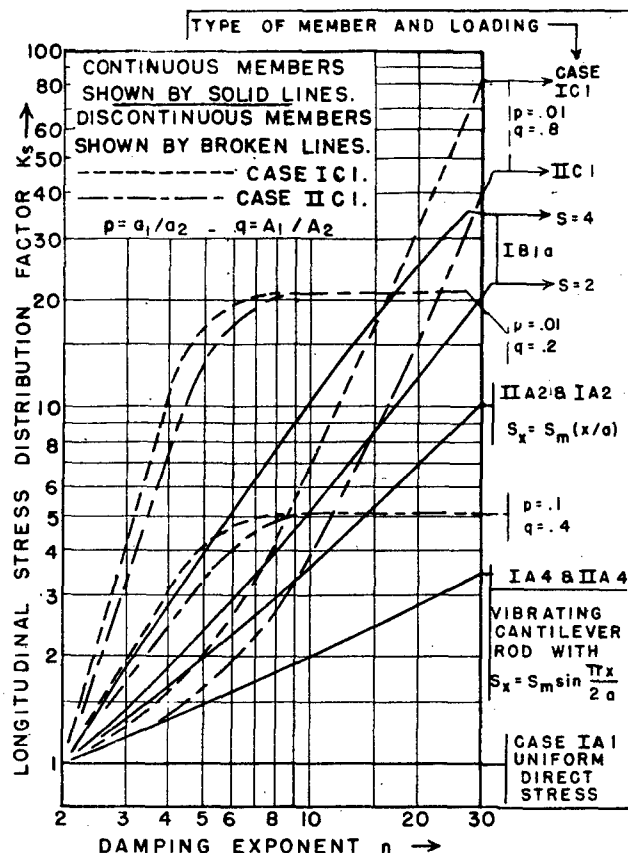

FIG. 2 LONGITUDINAL STRESS-DISTRIBUTION FACTOR K_s FOR VARIOUS COMMON TYPES OF MEMBERS AND LOADINGS AS A FUNCTION OF DAMPING EXPONENT n

resonance amplification factor A_r of the unnotched member, if other factors remain constant.

To illustrate the effect of damping exponent n on the cross-sectional shape factor K_c, three common sections were analyzed and diagrammed as shown in Fig. 3. The equations used to plot the characteristics of the diamond, circle, and rectangle are given in Table 2. The broken lines indicate the general trend for a modified diamond shape (with sharp upper and lower edges) and an I-beam. Since the exact shape of these sections governs the exact locations of the corresponding curves, these curves are qualitative indications only of the general trends.

It is significant to note from the curves that merely changing the cross-sectional shape of a beam from a diamond to an I-section (all other factors remaining constant) may decrease its resonance amplification factor very significantly, as much as 20 times under certain conditions. It is sometimes possible to make such shapes in cross-sectional shape in vacuum-tube components, structural beams, and other parts to reduce resonance amplification vibrations and increase service life without affecting functional behavior.

Referring again to the five shapes diagrammed in Fig. 3, it is apparent that the smaller the percentage volume at near peak stress, the larger is the K_c-factor, particularly at large values of damping exponent n. The physical reason for this mathematical observation may be explained on the basis of the ratio of the elastic energy to damping energy. In general, it may be stated that if either the cross-sectional shape or longitudinal stress distribution are such as to cause a large percentage of the material to be at stress low enough as to dissipate negligible damping energy, and yet at a stress high enough to store significant elastic
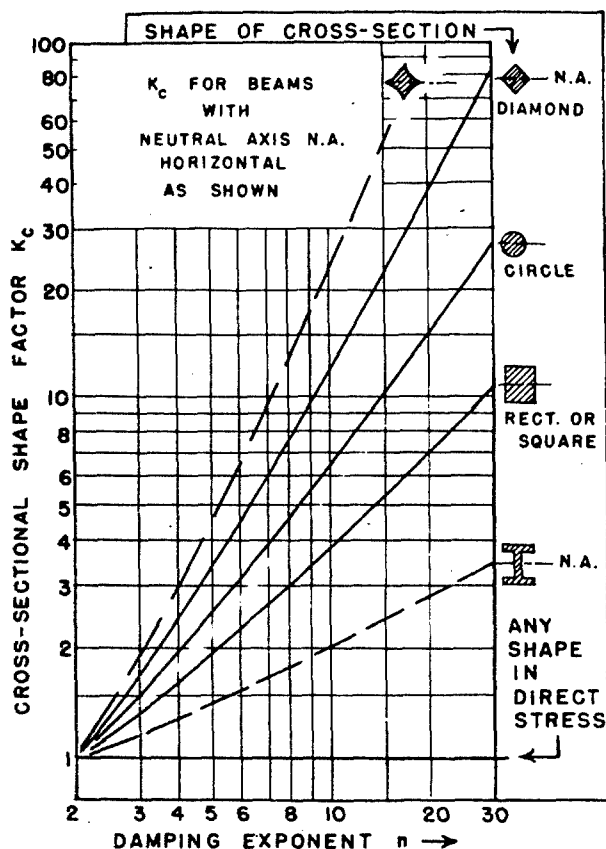
FIG. 3 CROSS-SECTIONAL SHAPE FACTOR K_c FOR VARIOUS COMMON SECTIONS AS A FUNCTION OF DAMPING EXPONENT n

energy, then relatively large A_r-factor will result. This is so for diamond-shaped cross section (as compared to I-beams), for tapered members at constant force (as compared to constant section members), for cantilever beams (as compared to constant moment beams), and for stepped or notched (discontinuous) members (as compared to continuous members). In general, as much volume as possible should be exposed to near peak stress if A_r is to be kept small.

An example of the method of using the three factors which govern the amplification in vibration caused by resonance is given in the following. The example considered is a cantilever beam of constant diamond cross-sectional shape made of S-816, vibrated at 900 F, and exposed to a maximum stress of 60,000 psi (virgin material assumed).

From Equation [16]

$$A_r = K_m K_s K_r$$

From Fig. 1, $K_m = 19$ for virgin material at 900 F, at 60,000 psi and damping exponent $n = 8$

From Fig. 2, $K_s = 1.7$ for a vibrating cantilever beam (case IIA4) with a damping exponent of 8

From Fig. 3, $K_c = 7.8$ for a diamond cross section of a material with a damping exponent of 8

$$A_r = 19 + 1.7 \times 718 = 252$$

If this resonance amplification factor is too large for the application under consideration, a change in cross section from diamond to I-section could be considered. The resultant reduction in K_c-factor from 7.8 to 1.7 would result in a reduction in resonance amplification from 252 to 55. Needless to say, the resultant

reduction in fatigue stress at resonance may be sufficient to eliminate service-failure difficulties in critically stressed members.

EXPERIMENTAL VERIFICATION FOR A_r-EQUATIONS

In order to procure at least partial experimental confirmation of the A_r equations derived in the previous sections a series of tests was undertaken with a newly developed resonance-vibration testing machine described in reference (4). With this machine it is possible to excite resonant vibrations in various types of systems and at the same time maintain the desired excited force by controlling the magnitude of the exciting force. Automatic controls maintain (a) a state of resonance and (b) the desired excited force, even under conditions of changing damping and dynamic stiffness in the system. The value of the resonance amplification factor A_r may then be computed from the ratio of the excited force (indicated by an accelerometer output) to the exciting force (determined from a tachometer and a counter which indicate unbalance in a revolving eccentric).

Tests were undertaken in this machine to determine A_r for a beam-type specimen made of alloy TP-2-B,[3] this alloy being selected because its large damping exponent n of 13 provides a more critical test of the theory than a material with a small damping exponent. In these tests, made at a stress of 80,000 psi and a stress history of approximately 1000 cycles, three cross-sectional shapes were used, i.e., a square with the neutral axis parallel to one side, a circle, and a diamond shape (actually the same specimen as used for the square cross-section test, except that it was oriented so that the neutral axis was diagonally across two opposite corners). The experimental A_r-values so determined and the theoretical A_r-values calculated by means of the equations presented previously are indicated in Table 3.

It might appear from Table 3 that the check between the theoretical and experimental values of A_r is rather poor. However, considering that in alloy TP-2-B (with a damping exponent n of 13) a 3 per cent error in stress magnitude will result in a 29 per cent error in K_m and A_r, and that the number of other variables which affect damping (such as variability of stress history and rest, cyclic stress frequency, machining stress, and others discussed in reference 3), the check between theory and experiment is considered adequate.

GENERAL OBSERVATIONS, SUMMARY, AND CONCLUSIONS

In many machines and structures, it is highly desirable, if not essential, that the amplitude of near-resonance vibration encountered in service be reduced. In general, the most satisfactory approach is, of course, to reduce the magnitude of the exciting force or change its frequency so as to avoid the resonant conditions. However, this cannot always be accomplished, in which case it becomes necessary either to make the machine more vibration-resistant or to increase the damping of the system in order to reduce the magnitude of the near-resonance vibrations. Frequently, the methods for increasing the fatigue and dynamic strength of a part are exhausted without a satisfactory solution, in which case one must resort to increasing damping.

Sometimes it is possible to increase the damping of a vibrating system through more effective use of joint, aerodynamic, and other damping not directly related to the material, shape, and stress distribution in the members. However, this too may be quite difficult to accomplish, in which case it becomes necessary to increase the damping of the actual part. It is this last type of damping which is the subject of this paper.

It was shown in this paper that the amplification A_r in vibration of a part at resonance is the product of three factors: (a) material factor K_m, (b) longitudinal stress-distribution factor K_s,

[3] See reference (5) for details on this material, including data on its damping, elasticity, and fatigue properties.

TABLE 3 COMPARISON OF THEORETICAL AND EXPERIMENTAL VALUES FOR FACTOR
A_r FOR TP-2-B AT A STRESS OF 80,000 PSI AND A STRESS HISTORY OF 1000 CYCLES
($D = 6.2 \times 10^{-63} S^{13}$: therefore, $J = 6.2 \times 10^{-33}$, and $n = 13$)

| Cross-sectional shape | Theoretically computed values | | | | Experimental A_r from resonance test |
	K_m see Eq. [8]	K_s see Eq. [9]*ᵃ	K_c see Fig. 3	A_r see Eq. [16]	
Square..........	14	2.2	4.8	150	170
Circle..........	14	2.2	9	280	340
Diamond........	14	2.2	18	550	950

ᵃ Tapered-beam specimens of the type used in rotating-beam damping work (see reference 3) to produce uniform stress distribution longitudinally also were used in these resonance tests. However, in the resonance tests the longitudinal stress distribution was not uniform. Therefore the K_s-factor for these specimens lies between 1 (for the case IIA1 having uniform longitudinal stress) and $(n + 1)/3 = 4.7$ (for case IIA2 having cantilever-beam-type stress). For the beam used K_s was computed to be very approximately 2.2.

and (c) cross-sectional shape factor K_c. The critical relationships of these factors to damping exponent n and other variables were derived and analyzed. Methods of decreasing the resonance amplification factor, based on the equations derived, were suggested and demonstrated. Experimental data were presented to verify the theory.

It is desirable at this point to generalize and interpret physically some of the mathematically derived K_s and K_c-relationships discussed in the previous sections. This is done to provide a guide for those who find it necessary to reduce the resonance amplification factor of parts which encounter troublesome resonance vibration.

Probably the most important generalizations and conclusions apparent from Tables 1 and 2, and Figs. 2 and 3, are as follows:

(a) For materials with low damping exponent, say, values of n near 2, the cross-sectional shape and longitudinal stress distribution in the members is relatively unimportant compared to the general level of material damping as defined by constant J.

(b) For materials with high damping exponent n, and many of the temperature-resistant materials may be so classified, the cross-sectional shape and longitudinal stress-distribution factors are extremely important. Relatively small changes in member shape and loading, which can sometimes be made through design without affecting functional behavior, cost, and so on, may greatly improve the ability of a part to withstand resonant conditions.

(c) As a guide for decreasing the K_s and K_c-factors for a part having large damping exponent n, an attempt should be made to design the part so that the percentage of material at peak or near-peak stress is as large as possible. Material at relatively low stress contributes a relatively large ratio of elastic energy to damping energy which tends to increase the resonance amplification factor.

It is felt that the analyses and observations discussed in this paper should provide the design engineer with improved perspective in dealing with problems involving resonant vibrations.

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